The role of examples in the learning of definite integral

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ABSTRACT: An investigation designed to reveal a range of responses regarding conceptual understanding of definite integral is presented and discussed in this article. The author gives an illustrated theoretical account of some roles for student-generated examples in the teaching and learning of mathematics. Students seem to have different structures of attention that dominate their understanding compared with experienced users. This study gives instances, which show that the development of example spaces contributes considerably to conceptual engagement. Students' responses to a single construction task reveal their awareness with regard to the understanding of the integral. Encouraging students to generate examples may help them gain a deeper understanding of a mathematical concept.

INTRODUCTION

The relationship between theories and examples is a basic issue in mathematics education in the transition from high school to university. At the beginning of undergraduate mathematics courses, a number of students are urged to turn from learning patterns based on prototypes and numerical algorithms to others that require handling general mathematical statements as well. Examples are used in a wide variety of ways in mathematical education as described for example in Bills et al [1] and references therein.

A major problem for higher education mathematics research is an understanding of the difficulties emerging from the nature of abstract concepts. In several questions where the rules, formulas and operations were successfully applied, the fact that students could not comprehend the mathematical ideas behind this process and relate them to different contexts has been a common problem for teachers and researchers [2]. This problem might emerge from the different problem-solving approaches of the students and the teachers [3].

Differentiation is a *forward* process; the difficulties faced by students in this concept are not as complicated as those in the reverse or *backward* process of integration. The many natures of integration: it is both the inverse process of differentiation and a tool for calculation of area and volume and length. The author became interested in the difficulties students face with calculus topics, particularly integration.

As a result, students may be well-equipped to work on the usual, textbook problems. However, when faced with questions that are slightly different, they can become incapacitated. In this article, the author reports on example generation to reveal some aspects of students' awareness of mathematical objects. Example generation may provide insight into the students' sense of generality and the constituents which comprise their understanding of the concept.

THEORETICAL FRAMEWORK

Giving examples of mathematical objects can be a complex task for students, as well as for teachers, but in spite of its complexity, such a task is rich in potentialities from an educational point of view [4]. In particular, asking students to construct their own examples is recognised as a powerful tool in mathematics education [5]. As suggested by Dahlberg and Housman *...it may be beneficial to introduce students to new concepts by requiring them to generate their own examples or have them verify and work with instances of a concept before providing them with examples and commentary* [6].

The relevance of learners' generating examples and the importance in mathematics education of the analysis of their processes are summarised in one of the research questions proposed by Bills et al: *...what is entailed and revealed by* *the process of constructing examples and how does construction of examples promote mathematical understanding?* [1]. Recently, some researchers highlighted the educational potentialities of the processes of example generation as a research tool, which *provides a* window *into a learner's mind* that can reveal and promote significant aspects of conceptualisation [7].

A number of researchers have considered the topic of integration. However, these researchers report on students' lack of flexibility, their inability to make necessary links/connections between concepts/ideas and their lack of understanding of underlying principles but without looking into the causes of such problems.

For example, Orton studied the understanding of integration of 110 students majoring in mathematics and explained how students have problems with the reasoning behind integration methods, particularly when calculating areas bounded by curves [8]. Ferrini-Mundy and Graham reported inconsistencies between performance on procedural items and conceptual understanding in that conflicting conceptions are held comfortably and routinely in the development of calculus concepts, with separate understandings for geometrical and algebraic contexts [9].

Students' preference for using procedural skills and their apparent reluctance to use geometric interpretations may explain their strong inclination to move to an algebraic context [10]. Norman and Prichard suggest that geometric intuitions about integration could become cognitive obstacles to the understanding of the concept [11]. Selden, Selden and Mason reported that even good calculus students often cannot solve non-routine problems [12]. Their study showed that students exhibited a tendency not to use calculus, preferring arithmetic and algebraic techniques for solving calculus problems, even though the use of elementary calculus methods would be more appropriate.

Conceptual understanding can be regarded not only as the ability to use situated knowledge to solve routine, textbook problems correctly but more importantly as the act of extending that situated knowledge appropriately and efficiently to unfamiliar situations. The use of examples to illustrate mathematical concepts has been an integral part of mathematics instruction. While the teacher uses examples to clarify definitions and exemplify use of certain rules, students may develop the thinking that only those kinds of examples are appropriate and not see the generalities behind the chosen examples. Coming up with examples, on the other hand, requires different cognitive skills from working out given examples. Dahlberg and Housman showed how students who generated examples and reflected on the process attained a more complete understanding of mathematical concepts by refining and expanding their evoked concept image [6].

Hazzan and Zazkis showed how students had difficulty managing degrees of freedom of generated examples [13]. Encouraging students to generate examples of mathematical objects can expand their *example space* and shift their attention away from the examples to generalisations [5]. Although students do not normally encounter this type of question in their learning of concepts, the process itself could bring out their ability to discern dimensions-of-possible-variation and reveal their awareness of the concept, which could reveal the structure of their understanding.

METHODOLOGY

The 15 first-year engineering students who participated in this study were enrolled at a university of technology and had learned the basic rules of integration using primitives, as well as their relationship to the calculation of a number of areas under curves. These students' calculus grades were in the top 10% among 352 students. The instruments used for data collection were a questionnaire containing problems and interviews. The questionnaire comprised five problems in definite integral. These problems enabled the students' performance regarding the example generation to be analysed.

The results of the questionnaire necessitated further investigation into the example generation of students. The clinical interviews were carried out after the answers to the problems had been analysed. Each interview lasted about 40-50 minutes and was video- and audio-taped. In order to prepare the script for the interview, the author analysed the written answers focused on how the students seemed to use and coordinate the different mathematical examples needed. During each interview the students were asked to think aloud, while they were solving the tasks so that the author could describe their responses and strategies, as well as make inferences about their examples.

The analysis focused on identifying the students' examples used to create meanings for the problems and the justifications provided. For each problem, the author identified:

- The examples that the student used;
- The relationships that the student established between the examples to generate new information.

This analysis provides information on how the student uses and relates the examples of mathematical knowledge to obtain new information.

ANALYSIS AND RESULTS

In constructing and generating examples, many respondents at first engaged in algebraic manipulation of the expression. John uses the method of change of variables to generate another example, while being investigated by the researcher (R):

R: Give me another example of $\int_0^2 (1-x)dx = 0$.
John: I could have just x limits -1 to 1, take the lower limit from the upper limit, so $\int_{-1}^{1} xdx$ makes zero.
R: Can you give me one more example?
John: Just the same way, let u-1 equal to 1-x, then changes $\int_0^2 (1-x)dx$ to $\int_2^0 (x-1)dx$, also makes zero.
R: Can you use the area under graph to generate some examples?
John: No.

Helen does not even use the method of change of variables. She provides examples, such as $\int_0^2 \frac{x}{x}(1-x)dx = 0$ and a general example of $\int_0^2 a(1-x)dx = 0$, realises that *a* can be anything that can be changed.

The students seemed to attempt this by simply putting in the values for limits into the expression, ignoring the $\int$ sign and being unaware of the fact that the definite integral is actually an area. They seemed to manipulate the example superficially. The integral sign $\int$ is seen as a command to do something and this turns into a command to plug in numbers. Students' awareness of imagery/associations (area under graph) seemed to be pushed to the background as they focused on techniques and were unaware of images of the object given as an example and the example they constructed.

There appears to be a preference to use equations, even in cases where a geometrical solution is more feasible. This is suggestive that many students lack flexibility of thought and do not seem to make the necessary connections to get a sense of the concept. When encountering integration problem, students tend to focus on technique alone and push down awareness of the concept in terms of imagery and links/associations. After further prompting, some of them did generate examples and came up with a general expression for the example after becoming aware of some dimensions of possible variation:

R: Can you tell me what functions make the integral $\int_{-a}^{a} f(x)dx$ equal to zero?
Tom: I take $\int_{-\pi}^{\pi} sinxdx$ that will come out to zero.
R: Why?
Tom: Because it is equal area on top and at the bottom.
R: Does $\int_{-3}^{3} x^5 dx$ equal to zero?
Tom: I have to evaluate the integral.
R: Why?
Tom: Because the graph of the function is difficult to draw.
R: OK.
Tom: The antiderivative is $1/6x^6$, so the integral is zero.
R: Give me another example.
Tom: I think that the integral $\int_{-3}^{3} x^7 dx$ also makes zero.
R: Why?
Tom: Because the antiderivative has even power, so the integral make 0.
R: Can you give me a general example?
Tom: I see, if the integrand has odd power, then the integral must be zero. like $\int_{-a}^{a} x^{odd} dx$.

Tom knows the relationship between definite integral and area, but he has to draw the graph of function to justify the area above x-axis and the area below x-axis. Porter is another example; he can generate more general examples:

R: Give me another example.
Porter: The integral $\int_{-\pi/4}^{\pi/4} tanxdx$ come out to zero.
R: Why?
Porter: One way is to evaluate the integral, the other way is to examine the area, area above x-axis is equal to the area below x-axis, so the integral must be zero.
R: Can you give me another example?
Porter: The integral $\int_{-3}^{3} x^7 dx$ is zero.
R: Why?
Porter: Because the antiderivative has even power, then you have the same positive number on either side of the $\int$ sign that should always come up to zero.
R: Good, can you give me a general example?
Porter: The integrand must be odd power, so the antiderivative has even power, $\int_{-a}^{a} x^{2n+1} dx$ is zero.
R: Good, can you give me a more general example?
Porter: $\int_{-a}^{a} kx^{2n+1} dx$ or $\int_{-a}^{a} (px^{2n+1} + qx^{2m+1}) dx$.

In generating examples, the students first attempted to change the numbers only, but by generating more examples, they realised they could change things that are variable and maintain that, which is not permitted to change according to the criteria/situation. The students seemed confused as they tried to search for accessible objects and, then, constructed new examples from available information.

However, for Task 3, Helen says that the proposition was true and provided specific examples of functions as evidence without giving graphic representations, failing to provide suitable justifications. She provided a specific example that defined two functions $f(x) = x^2+1$ and $g(x) = x^2$, and calculated two integrals between x = 1 and x = 2 to obtain 10/3 for f and 7/3 for g. Subsequently, the interview progressed as shown below:

R: Can you provide a geometric or numerical example?
Helen: (Draws the two curves of $f(x) = x^2+1$ and $g(x) = x^2$) Like this?
R: Can you do this in graph form without an equation?
Helen: How do I draw graphs without equations?
R: OK, can you draw graphs that represent the definite integral?
Helen: No, I do not know how to do it.

For Task 3, Kevin stated that the proposition was false and provided graphical representations (Figure 1). Subsequently, the interview progressed as shown below:

R: If f(x) is greater than g(x), would the integral of f(x) be greater than the integral of g(x)?
Kevin: Yes.
R: Why?
Kevin: If f(x) is greater than g(x), the difference of f(x) minus g(x) would be greater than 0 and a positive value. The integral of f(x) minus g(x) would be a positive value; therefore, the integral of f(x) would be greater than the integral of g(x).
R: Why do you think Task 3 is incorrect?
Kevin: The situation in Task 3 is opposite to that of your question. The integral values of the function in the interval [a, b] are greater, and the values of the function in the interval [a, b] are not necessarily greater than that of g(x).
R: Why is that?
Kevin: In this graph I drew, the area below f is greater than the area below g in the interval [a, b]; therefore, the integral of f in [a, b] is greater than the integral of g in [a, b]. However, the function value of f in the interval [a, c] is smaller than the function value of g in the interval [a, c]. Therefore, the description in this task is incorrect.

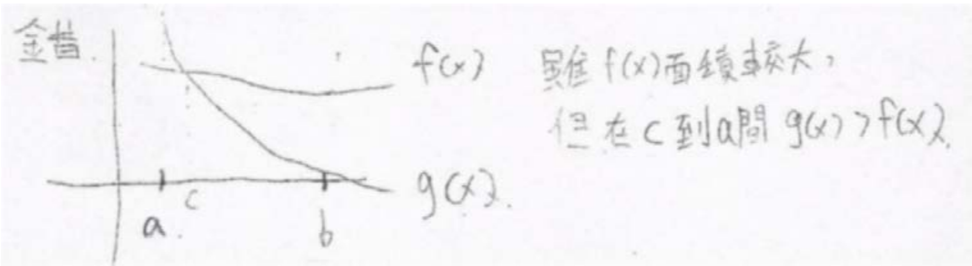

Figure 1: Kevin's graphical presentation for Task 3.

CONCLUSIONS

Example construction tasks offer possibilities for students to reveal their conceptual understanding of mathematical ideas and to enrich their repertoire of examples. Once students were prompted to produce more examples and recognised the freedom this offered, they generalised at least to some extent. The responses reveal some awareness of things that can vary but students tended to maintain the form of the original function. Realisation of things that can vary, and more importantly, exercising of their freedom to change properties of mathematical objects can, indeed, educate students' awareness.

Generating examples can not only enrich example space in terms of its content, but it can also provide a chance for students to explore its structure in terms of relationships among elements in the space, which in turn can reveal and alter students' sense of generality. Thus, it is important that students do not simply engage in tasks in order to get answers but to appreciate their autonomy and, thus, educate their awareness. Teachers must be explicit with their emphasis on structural properties of examples so that students can see the general through the particular. Unless they are provided with such opportunities, students are likely to be confined to ritualised habits of rote learning mathematics.

The results presented in this article confirm the relevance of the role of pivotal-bridging examples as already observed by Zazkis and Chernoff [14]. Such examples should be constructed carefully, analysing their different linguistic and semiotic aspects with particular attention, to evoke and possibly resolve cognitive conflicts in the concept image of the students. It is the author's opinion that the power of these examples is also testified by the fact that they tend to impress in the memory of students, because of their peculiar features, thus further helping their learning process. Teaching patterns should, then, be conceived, including and taking full advantage of such pivotal-bridging examples. Anyway, it cannot be concealed that pivotal-bridging examples seem to work more effectively for students with reasonable linguistic and mathematical skills.

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